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# $p$-adic Cohomology

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The purpose of this paper is to survey some recent results in the theory of "$p$-adic cohomology", by which we will mean several different (but related) things: the de Rham or $p$-adic étale cohomology of varieties over $p$-adic fields, or the rigid cohomology of varieties over fields of characteristic $p > 0$. Our goal is to update Illusie's beautiful 1994 survey [**I**] by reporting on some of the many interesting results that postdate it. In particular, we concentrate more on the present and near-present than the past; [**I**] provides a much better historical background than we can aspire to, and is a more advisable starting point for newcomers.

Before beginning, it is worth pointing out one (but not the only) crucial reason why so much progress has been made since the appearance of [**I**]. In the mid-1990s, it suddenly became possible to circumvent the resolution of singularities problem in positive characteristic thanks to de Jong's alterations theorems [**dJ1**], which provide forms of "weak resolution" and "weak semistable reduction". (The weakness is the introduction of an unwanted but often relatively harmless finite extension of the function field: for instance, an alteration is a morphism which is proper, dominant, and generically finite rather than generically an isomorphism.) The importance of de Jong's results cannot be overstated; they underlie almost every geometric argument cited in this paper! See [**Brt6**] for more context regarding alterations.

## 1. Rigid cohomology

For concepts in rigid analytic geometry which we do not explain here, we recommend [**FvdP**] (or for more foundational details [**BGR**]).

**1.1. Preview: Monsky-Washnitzer cohomology.** Let $k$ be a field of characteristic $p > 0$, and let $K$ be a complete discretely valued field of characteristic 0 with residue field $k$ and ring of integers $\mathfrak{o}$, equipped with a lift of some power of the absolute Frobenius on $k$. For instance, one may take $k$ perfect and $K = \operatorname{Frac} W(k)$, in which case the lift of Frobenius exists and is unique.

Monsky and Washnitzer [**MW, Mon1, Mon2**] (see also [**vdP**]) described a cohomology theory for smooth affine $k$-varieties,[1] which they called "formal cohomology". Suppose $A = \mathfrak{o}[x_1, \dots, x_n]/I$ is a smooth affine $\mathfrak{o}$-algebra. Define the ring

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[1]For simplicity, here "$k$-variety" will mean "reduced separated scheme of finite type over $k$"; some results cited do not actually depend on the reduced or separated hypotheses.

of *overconvergent power series* $\mathfrak{o}\langle x_1, \dots, x_n\rangle^\dagger$ as the union, over all $\rho > 1$, of the ring of power series in $\mathfrak{o}[\![x_1, \dots, x_n]\!]$ which converge for $|x_1|, \dots, |x_n| \leq \rho$; put

$$A^\dagger = \mathfrak{o}\langle x_1, \dots, x_n\rangle^\dagger / I\mathfrak{o}_K\langle x_1, \dots, x_n\rangle^\dagger.$$

The Monsky-Washnitzer cohomology of $\mathrm{Spec}(A \otimes_{\mathfrak{o}} k)$ is then the cohomology of the complex of continuous differentials of $A^\dagger \otimes_{\mathfrak{o}} K$.

The bad news about this construction is that the above description leaves many questions unanswered: whether a smooth $k$-variety can be lifted to $\mathfrak{o}$ (so that the above construction is possible), whether it is independent of the choice of lift, whether maps between smooth $k$-varieties can be lifted, and whether the induced maps on cohomology are functorial (and in particular independent of the choice of lift). All of these have affirmative answers, but with a bit of work required; see [**vdP**].

The good news is how simple the construction is to describe. This makes it the centerpiece of much of the theoretical analysis of rigid cohomology, but it also has a quite unexpected side benefit: the construction has attracted much interest from computational applications that require the determination of the zeta function or related information[2] from a given variety over a finite field. For instance, this occurs in cryptography based on elliptic or hyperelliptic curves. The idea to use $p$-adic cohomological methods for such computations is due to Wan [**LW, Wa**]; see [**Ke1**] for a survey of some subsequent developments.

**1.2. Construction of rigid cohomology.** Monsky-Washnitzer cohomology can be thought of as an analogue of algebraic de Rham cohomology for smooth affine varieties. Berthelot [**Brt2**] (also see [**Brt4**]) realized that this should be generalized to an analogue of the algebraic de Rham cohomology of arbitrary varieties, constructed (following Herrera-Lieberman [**HL**] and Hartshorne [**Har1, Har2**]) by locally embedding a given variety into a smooth $k$-variety which lifts to $\mathfrak{o}_K$.

To be specific, suppose that $X$ is a $k$-variety, $X \hookrightarrow Y$ is an open immersion of $k$-varieties with $Y$ proper over $k$, and $Y \hookrightarrow P_k$ is a closed immersion for $P$ a smooth formal $\mathfrak{o}_K$-scheme. Let $P_K$ denote the Raynaud generic fibre of $P$; its points correspond to integral formal subschemes of $P$ which are finite flat over $\mathfrak{o}_K$. In particular, $P_K$ admits a specialization map $\mathrm{sp} : P_K \to P_k$; for a subset $U$ of $P_k$, we write $]U[$ for $\mathrm{sp}^{-1}(U)$ and call it the *tube* of $U$ in $P_K$.

A *strict neighborhood* of $]X[$ in $]Y[$ is an admissible open subset of $]Y[$ which together with $]Y \setminus X[$ forms an admissible covering of $]Y[$. The *rigid cohomology* $H^i_{\mathrm{rig}}(X/K)$ of $X$ is then constructed as the direct limit of the de Rham cohomologies of strict neighborhoods of $]X[$ in $]Y[$. There is a related but slightly more complicated construction of *rigid cohomology with compact supports* $H^i_{c,\mathrm{rig}}(X/K)$.

As for Monsky-Washnitzer cohomology, one must make some laborious calculations to verify that the construction of rigid cohomology with and without supports are independent of choices and appropriately functorial. We note in passing that

[2]The "related information" is often the order of a Jacobian group, but sometimes not. One example: Mazur-Stein-Tate [**MST**] use the matrix of the Frobenius action on $p$-adic cohomology to compute $p$-adic global canonical heights on elliptic curves over $\mathbb{Q}$. Another example: Voloch and Zarzar [**VZ**] use upper bounds on Picard numbers of surfaces to construct good error-correcting codes.

it would simplify foundations[3] to have either a description of rigid cohomology either in terms of cohomology on an appropriate site, or a suitable de Rham-Witt complex; these have been developed by le Stum [**LS**] and Davis, Langer, and Zink [**DLZ**], respectively.

Substituting de Jong's alterations theorem for resolution of singularities in a program suggested in [**I**, §4.3], and using a comparison theorem between rigid cohomology and crystalline cohomology (the latter being described in [**Brt1, BO1**]), Berthelot [**Brt4, Brt5**] established the following results.

(a) The vector spaces $H^i_{\mathrm{rig}}(X/K)$ for $X$ smooth, and $H^i_{c,\mathrm{rig}}(X/K)$ for $X$ arbitrary, are finite dimensional over $K$

(b) For $X$ smooth of dimension $d$, there is a perfect Poincaré duality pairing

$$H^i_{\mathrm{rig}}(X/K) \times H^{2d-i}_{c,\mathrm{rig}}(X/K) \to K.$$

(c) For $X_1, X_2$ smooth, there is a Künneth decomposition

$$H^i_{\mathrm{rig}}(X_1 \times_k X_2/K) \cong \oplus_j H^j_{\mathrm{rig}}(X_1/K) \otimes_K H^{i-j}_{\mathrm{rig}}(X_2/K);$$

for $X_1, X_2$ arbitrary, the analogous decomposition holds for cohomology with compact supports.

Finite dimensionality for general $X$ was deduced (using Berthelot's work) by Grosse-Klönne [**GK2**] using "dagger spaces" as introduced in [**GK1**]; these are a rigid analogue of Meredith's weak formal schemes [**Mer**].

In addition, the existence and basic properties of cycle class maps have been established by Petrequin [**P**]. This means that rigid cohomology is indeed a Weil cohomology in the sense of Kleiman [**Kl**].

**1.3. Overconvergent $F$-isocrystals.** A natural next step after establishing the basic properties of rigid cohomology is to look for an appropriate category of coefficient objects. One natural category are the *convergent $F$-isocrystals*;[4] in the notation of the previous section, these are coherent modules with connection on the tube $]X[$ which induce Taylor isomorphisms between the two pullbacks to the tube of $X$ in $(P \times_{\mathfrak{o}_K} P)_K$, equipped with Frobenius action.

To obtain better cohomological properties,[5] we must restrict attention to the *overconvergent $F$-isocrystals*. These are coherent modules with connection on a strict neighborhood of the tube $]X[$, which induce isomorphisms between the two pullbacks to a strict neighborhood of the tube of $X$ in $(P \times_{\mathfrak{o}_K} P)_K$, again equipped with Frobenius action. There is a natural faithful functor from overconvergent $F$-isocrystals to convergent $F$-isocrystals; on smooth varieties,[6] this functor is fully faithful [**Ke4**]. The rigid cohomology of $X$ with coefficients in an overconvergent

[3] For instance, either of these descriptions might make it easier to consider rigid cohomology for algebraic stacks, which should compare to the crystalline cohomology for algebraic stacks described in [**Ol1**]. For simplicity, we withhold any further discussion of stacks from this paper.

[4] This somewhat awkward name, and its "overconvergent" sibling, deserve some clarification. The "convergence" here is of the Taylor series isomorphism; the $F$ denotes the Frobenius action; the "isocrystal" is short for "crystal up to isogeny", which is how these objects first arose in the work of Berthelot and Ogus [**BO2, Og1, Og2**].

[5] One way to envision the difference between convergence and overconvergence from a geometric viewpoint, following a suggestion of Daqing Wan, is that convergent and overconvergent $F$-isocrystals correspond to motives with coefficients in finite extensions of $\mathbb{Z}_p$ and $\mathbb{Z}$, respectively.

[6] The same result for general varieties can probably be deduced using a descent argument, but this does not seem to have been verified.

$F$-isocrystal can be defined, in the local situation, as the direct limit of the de Rham cohomology of the connection module over strict neighborhoods of $]X[$; again, there is a similar definition of cohomology with compact supports.

The rigid cohomology of overconvergent $F$-isocrystals on curves was closely analyzed by Crew [**Cr2**], who proved finite dimensionality and Poincaré duality, under a certain hypothesis which has since been verified; see Section 1.4. One interesting aspect of Crew's work is its blend of ideas from algebraic geometry and functional analysis, which played a crucial role in subsequent developments.

Based on Crew's work, Kedlaya[7] [**Ke6**] established analogues of Berthelot's finiteness, Poincaré duality, and Künneth formula results for overconvergent $F$-isocrystals. Again it takes more work to obtain finiteness of cohomology without supports on nonsmooth schemes; for this one needs cohomological descent for proper hypercoverings, developed by Chiarellotto-Tsuzuki [**CT, Tsz5, Tsz6**].

Some other results have been successfully analogized into rigid cohomology. The Grothendieck-Ogg-Shafarevich formula, expressing Euler characteristics of overconvergent $F$-isocrystals on curves in terms of certain Swan conductors, follows from a local index theorem of Christol-Mebkhout [**ChM4**] and a "Swan conductor equals irregularity"[8] theorem of Crew [**Cr3**], Matsuda [**Mat**], and Tsuzuki [**Tsz2**]. The Lefschetz formula for Frobenius in rigid cohomology was established by Étesse and le Stum [**ElS**]. An analogue of Deligne's "Weil II" purity theorem, using a version of Laumon's Fourier transform (see Section 2.1), was established by Kedlaya [**Ke7**], building on work of Crew [**Cr1, Cr2**]; an extension to certain complexes of arithmetic $\mathcal{D}$-modules (see Section 2.1) was given by Caro [**Cro2**].

One can ask a seemingly limitless number of further questions asking for an analogue of a given result in étale cohomology; most of these remain as yet unconsidered. One particularly intriguing one is Mokrane's analogue [**Mok**] of the weight-monodromy conjecture in étale cohomology; it might be possible to interpret this question in terms of rigid geometry. If one can relate the problem to Grosse-Klönne's dagger spaces, at least in the setting of semistable reduction (which is specially treated in [**GK2**]), one might obtain results stronger than, not just equal to, those known in étale cohomology. Such a geometric interpretation has already been given for curves by Coleman and Iovita [**CI1, CI2**].

**1.4. Local monodromy in rigid cohomology.** In this section, we expand on the notion of "local monodromy" in rigid cohomology, which underpins most of the results on overconvergent $F$-isocrystals cited in the previous section. It also exhibits a surprising link with $p$-adic Hodge theory; see Section 3.2. A nice discussion of this topic is given by Colmez [**Cmz1**].

In Crew's analysis [**Cr2**] of the cohomology of an overconvergent $F$-isocrystal on a curve, one is led to consider what amounts to local cohomology at the missing points. A missing point[9] over $k$ lifts to an open unit disc over $K$, and the $F$-isocrystal only extends to an unspecified annulus near the boundary of that disc. One thus naturally obtains a finite free module over the *Robba ring* $\mathcal{R}_K$ of series

[7] This argument is an exception to our initial comment about de Jong's alterations theorem; here one also uses a geometric argument [**Ke5**] based on a higher-dimensional analogue of "Abhyankar's trick", a strong form of Belyi's theorem in positive characteristic.

[8] The analogy between irregularity and Swan conductors has also been pointed out in the setting of complex analytic de Rham cohomology, by Bloch and Esnault [**BE**].

[9] One can make a similar analysis for non-rational closed points, which we omit for simplicity.

$\sum_{n \in \mathbb{Z}} c_n t^n$ convergent on some unspecified annulus $\eta < |t| < 1$, equipped with commuting Frobenius and connection actions.

This setup led Crew to propose (modulo a later refinement by Tsuzuki [**Tsz3**] and a reformulation by de Jong [**dJ2**]) the following then-conjectural statement. Let $\mathcal{R}_K^{\mathrm{inte}}$ denote the subring of $\mathcal{R}_K$ of series with coefficients in $\mathfrak{o}_K$; this is a noncomplete but henselian discrete valuation ring.

THEOREM 1.1 (Local monodromy theorem in rigid cohomology). *Let $M$ be a finite free module over the Robba ring $\mathcal{R}_K$, equipped with commuting Frobenius and connection actions. Then $M$ admits a filtration stable under the actions, whose successive quotients become trivial connection modules after tensoring over $\mathcal{R}_K^{\mathrm{inte}}$ with some unramified extension.*

Since there are two essential structures attached to $M$, the Frobenius and connection actions, it is fitting that there are two approaches to proving Theorem 1.1 emphasizing the two structures. One approach uses the theory of $p$-adic differential equations initiated by Dwork and Robba, specifically the $p$-adic local index theory of Christol-Mebkhout [**ChM1, ChM2, ChM3, ChM4**]. That theory makes essentially no reference to Frobenius actions; it is combined with further analysis in distinct ways by André [**A**] and Mebkhout [**Meb2**] to give two proofs of Theorem 1.1.

The other approach, following a strategy suggested by Tsuzuki [**Tsz3**], is to develop a structure theorem for Frobenius actions on the Robba ring, related to the Dieudonné-Manin classification of rational Dieudonné modules. Such a theorem is given by Kedlaya [**Ke3**] (see also [**Ke6**] for a simplified presentation); it combines with an analysis of Theorem 1.1 in the case of "unit-root Frobenius" due to Tsuzuki [**Tsz1**] to again yield Theorem 1.1.

It is an interesting problem to extend Theorem 1.1 to a sensible notion of local monodromy for overconvergent $F$-isocrystals on higher-dimensional spaces. The "generic local monodromy theorem" of [**Ke6**] makes a first run at this; a more complete answer would follow from resolution of a conjecture of Shiho [**Sh2**], which asserts that any overconvergent $F$-isocrystal can be pulled back along a suitable alteration to obtain something extendable to a log-isocrystal on a proper variety. For more discussion of this question, see [**Ke8**] (where it is referred to as the "semistable reduction problem" for overconvergent $F$-isocrystals).

## 2. Arithmetic $\mathcal{D}$-modules

See Berthelot's excellent survey [**Brt8**]. for a more detailed description of most of what we explain here, except for Caro's work which postdates [**Brt8**].

**2.1. $\mathcal{D}$-modules.** The category of overconvergent $F$-isocrystals, while useful in many ways, suffers the defect of not supporting Grothendieck's six operations; that is because this category is only analogous to the category of lisse (smooth) sheaves in $\ell$-adic cohomology. A proper category of coefficient objects would also include analogues of constructible sheaves; in de Rham cohomology, this is accomplished by considering the desired coefficient objects to be modules for a typically noncommutative ring of differential operators, as in the work of Bernstein, Kashiwara, Mebkhout, etc.

Motivated by this last consideration (and by some initial $p$-adic constructions of Mebkhout and Narváez-Macarro [**MN**]), Berthelot [**Brt3, Brt7**] has introduced a notion of *arithmetic* $\mathcal{D}$*-modules*. In the smooth liftable case these admit a "Monsky-Washnitzer" description: suppose $\mathfrak{X}$ is a smooth formal scheme over $\mathfrak{o}_K$, and let $\mathcal{D}_{\mathfrak{X}}$ be the usual ring sheaf of algebraic differential operators on $\mathfrak{X}$ (with respect to $\mathfrak{o}_K$, but to lighten notation we suppress this). For $m$ a positive integer, if $x_1, \dots, x_n$ are local coordinates, then the ring subsheaf of $\mathcal{D}_{\mathfrak{X}}$ generated by the operators $\frac{1}{(p^i)!}\frac{\partial^{p^i}}{\partial x_j^{p^i}}$ for $i \leq m$ and $j = 1, \dots, n$ does not depend on the choice of coordinates; the description thus sheafifies to give a ring subsheaf $\mathcal{D}_{\mathfrak{X}}^{(m)}$ of $\mathcal{D}_{\mathfrak{X}}$. Let $\widehat{\mathcal{D}}_{\mathfrak{X}}^{(m)}$ be the $p$-adic completion of $\mathcal{D}_{\mathfrak{X}}^{(m)}$, and put

$$\mathcal{D}_{\mathfrak{X}}^{\dagger} = \bigcup_{m=1}^{\infty} \widehat{\mathcal{D}}_{\mathfrak{X}}^{(m)}, \qquad \mathcal{D}_{\mathfrak{X},\mathbb{Q}}^{\dagger} = \mathcal{D}_{\mathfrak{X}}^{\dagger} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Berthelot showed that $\mathcal{D}_{\mathfrak{X},\mathbb{Q}}^{\dagger}$ is a coherent sheaf of rings, and that its category of coherent modules is functorial in the special fibre $X = \mathfrak{X}_k$; this category includes the convergent isocrystals on $X$.

In practice, one proves foundational results about objects associated to $\mathcal{D}_{\mathfrak{X},\mathbb{Q}}^{\dagger}$, like the derived category $D^b_{\mathrm{coh}}(\mathcal{D}_{\mathfrak{X},\mathbb{Q}}^{\dagger})$ of bounded complexes of $\mathcal{D}_{\mathfrak{X},\mathbb{Q}}^{\dagger}$-modules with coherent cohomology, by writing them as direct limits of related objects over the $\widehat{\mathcal{D}}_{\mathfrak{X}}^{(m)}$ and proving the statements there by working modulo a power of $p$. For instance, this is how one constructs the standard cohomological operations for $\mathcal{D}$-modules, namely internal and external tensor product, direct and inverse image, and exceptional direct and inverse image [**Brt8**, §4.3]. It is also how one establishes the $\mathcal{D}$-module version of Serre duality, in this setting due to Virrion [**V1, V2, V3, V4, V5**]; its compatibility with Frobenius is due to Caro [**Cro3**].

**2.2. Overconvergent singularities.** One might think that since $\mathcal{D}_{\mathfrak{X},\mathbb{Q}}^{\dagger}$ is a ring of overconvergent differential operators, we should be able to use it to talk about overconvergent isocrystals. That is not the case, though, because $\mathcal{D}_{\mathfrak{X},\mathbb{Q}}^{\dagger}$ is only "overconvergent in the differential direction" and not in the "coordinate direction" (since in particular $\mathcal{O}_{\mathfrak{X}} \subset \mathcal{D}_{\mathfrak{X},\mathbb{Q}}^{\dagger}$). To fix this,[10] one allows consideration of differential operators with overconvergent singularities along a divisor, as follows. (For a more detailed description, see [**Brt8**, §4.4].)

Let $Z$ be a divisor on $X = \mathfrak{X}_k$. Since we are giving a local description, we will assume $\mathfrak{X} = \operatorname{Spf} A$ is affine and that $Z$ is cut out within $X$ by the reduction of some $f \in \Gamma(\mathcal{O}_{\mathfrak{X}}, \mathfrak{X})$. Let $\widehat{\mathcal{B}}_{\mathfrak{X}}^{(m)}(Z)$ be the ring sheaf corresponding to the completion of $A[T]/(f^{p^{m+1}}T - p)$. Change the subscript $\mathfrak{X}$ to $\mathfrak{X}, \mathbb{Q}$ to denote tensoring with $\mathbb{Q}$ over $\mathbb{Z}$. Put

$$\mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^{\dagger}Z) = \varinjlim_{m} \widehat{\mathcal{B}}_{\mathfrak{X},\mathbb{Q}}^{(m)}(Z);$$

this is the sheaf of functions on $\mathfrak{X}$ with overconvergent singularities along $Z$. An important result is that $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^{\dagger}Z)$ is a coherent $\mathcal{D}_{\mathfrak{X},\mathbb{Q}}^{\dagger}$-module [**Brt8**, Théorème 4.4.7].

[10] One could presumably also fix this by redoing the theory with $\mathfrak{X}$ taken to be a Meredith weak formal scheme. Indeed, this is the original approach of Mebkhout and Narváez-Macarro in [**MN**]; however, Berthelot's approach is better suited for relating the theory to the rigid cohomology of nonsmooth varieties.

To get a sheaf of overconvergent differential operators with overconvergent singularities, we take a direct limit of completed tensor products: namely, define

$$\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}({}^\dagger Z) = \varinjlim_m (\widehat{\mathcal{B}}^{(m)}_{\mathfrak{X}}(Z) \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

One can carry over the study of cohomological operations and duality to this setting; see the Virrion and Caro references above.

For $\mathfrak{X}$ proper, $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger Z)$ gives a sheafified version of the Monsky-Washnitzer algebra associated to $U = X \setminus Z$. This gives a way to equip an overconvergent isocrystal on $U$ with the structure of a $\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}({}^\dagger Z)$-module, whose cohomology is directly related to the rigid cohomology of the isocrystal. This $\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}({}^\dagger Z)$-module is in fact coherent, but this requires a nontrivial argument, given by Caro [**Cro4**].

**2.3. Fourier transforms.** One important component of the study of arithmetic $\mathcal{D}$-modules is the Fourier transform; it analogizes both the natural Fourier transform in the algebraic and analytic $\mathcal{D}$-module settings (more on which shortly) and the geometric Fourier transform introduced by Deligne and Laumon in étale cohomology. In this context, the Fourier transform was constructed by Huyghe in her thesis [**Hu1**], the contents of which appear in a series of papers [**Hu2, Hu3, Hu4, Hu5, Hu6, Hu7**].

The Fourier transform[11] is easiest to describe on the affine line, so let us do that now. Assume that $K$ contains a chosen root $\pi$ of the equation $\pi^{p-1} = -p$. Take $\mathfrak{X} = \mathbb{P}^1_{\mathfrak{o}_K}$, $Z$ to be the point at infinity, and let $x$ be the coordinate on $\mathbb{A}^1_K$. Let $\mathcal{L}_\pi$ denote the *Dwork isocrystal* on $\mathbb{A}^1_k$: it corresponds to a $\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}(\infty)$-module which over $\mathcal{O}_{\mathfrak{X},\mathbb{Q}}({}^\dagger Z)$ is free of rank 1 generated by $\mathbf{v}$, such that the action of $\frac{d}{dx}$ on $\mathbf{v}$ is multiplication by $\pi$. Then the Fourier transform on $\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}$-modules is defined by appropriately interpreting the instruction: "Construct an integral operator with kernel $\mathcal{L}_\pi$." Loosely, one pulls back from $\mathfrak{X}$ to one factor of $\mathfrak{X} \times \mathfrak{X}$, tensors with the pullback of $\mathcal{L}_\pi$ along the multiplication map $\mathbb{A}^1 \times \mathbb{A}^1$, then pushes forward along the other factor. (The reality is a bit more complicated because one must work on $\mathbb{P}^1$ and not $\mathbb{A}^1$, where there is a bit of blowing up involved. See [**Hu1**, §4] or [**Hu7**, §3].)

This construction happily admits an alternate description which clarifies some of its properties. Write $\partial^{[i]}$ for $\frac{1}{i!}\partial^i$. Let $A(K)$ denote[12] the set of formal sums $\sum_{i,j=0}^\infty c_{i,j} x^i \partial^{[j]}$ such that the ordinary power series $\sum_{i,j=0}^\infty c_{i,j} x^i \partial^j$ belongs to $K\langle x, \partial\rangle^\dagger$. In fact, $A(K)^\dagger$ is the set of global sections of $\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}({}^\dagger Z)$; in particular, $A(K)^\dagger$ admits a noncommutative ring structure in which $\partial x - x\partial = 1$. Moreover, any coherent $\mathcal{D}^\dagger_{\mathfrak{X},\mathbb{Q}}({}^\dagger Z)$-module is represented by a coherent $A(K)^\dagger$-module [**Hu3**]. In this notation, the Fourier transform of a coherent $A(K)^\dagger$-module is given, up to a shift in degree, by pullback by the map $A(K)^\dagger \to A(K)^\dagger$ defined by $x \mapsto -\partial/\pi$, $\partial \mapsto \pi x$ [**Hu4**].

The Fourier transform for arithmetic $\mathcal{D}$-modules is expected to have the same sorts of uses as in étale cohomology. At least one of these has already been realized:

[11] We are describing the Fourier transform without supports; there is also a Fourier transform with compact supports, but Huyghe [**Hu1**] showed using duality that as in the analogous settings, the two transforms coincide.

[12] In Huyghe's notation, this ring would be denoted $A_1(K)$, as it has an $n$-dimensional analogue $A_n(K)$.

Mebkhout [**Meb1**] suggested using a $p$-adic Fourier transform on the affine line to implement Laumon's proof of Deligne's "Weil II" in $p$-adic cohomology, and this has been done (see Section 1.3).

**2.4. Holonomicity: unfinished business.** The concept of an arithmetic $\mathcal{D}$-module is something of a hybrid, sharing characteristics both of the complex analytic and the algebraic analogues. In particular, for a coherent $\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}}$-module equipped with an action of absolute Frobenius (i.e., a coherent $F$-$\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}}$-module), one can define the characteristic cycle, prove an analogue of Bernstein's inequality, and define a *holonomic* $F$-$\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}}$-module as one in which equality holds in Bernstein's inequality. However, the definition of holonomicity cannot be made as in the algebraic case because we are working with differential operators of infinite order, and it cannot be made as in the analytic case because we do not[13] have an analogue of pseudodifferential operators and their symbols. Instead, the notion of holonomicity is defined using a process of Frobenius descent: the presence of a Frobenius structure makes it possible to descend a module from $\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}}$ to $\widehat{\mathcal{D}}^{(m)}_{\mathfrak{X}}$ for some $m$. One can then descend further to remove the completion, finally ending up in an algebraic situation where one can resort to the usual concept of a good filtration in order to define the characteristic variety. This process is described in detail in [**Brt8**, §5].

The good news is that this notion of holonomicity leads to a natural proposal for a category of coefficients in $p$-adic cohomology (by taking the derived category of bounded complexes with holonomic cohomology plus some restriction on supports). The bad news is that the indirect nature of the definition of holonomicity makes it extremely difficult to verify the stability of holonomicity under the cohomological operations! As a result, Berthelot's program to construct a good $p$-adic coefficient theory has been stalled; see [**Brt8**, §5] for several conjectures which await resolution.

A possible route around this difficulty has been proposed by Caro [**Cro5**], who defines a category of *overholonomic* $F$-$\mathcal{D}^{\dagger}_{\mathfrak{X},\mathbb{Q}}$-modules, by building into the definition a certain stability of coherence under smooth base change, and the ability to perform dévissages in overconvergent $F$-isocrystals in the manner of [**Cro2**]. Caro shows that this category is stable under all of the cohomological operations except internal and external tensor product. He also shows that Berthelot's conjectures imply the coincidence between the notions of holonomicity and overholonomicity (and of a related notion of "overcoherence" introduced in [**Cro1**]). Moreover, he shows that unit-root overconvergent $F$-isocrystals are overholonomic, as are arbitrary overconvergent $F$-isocrystals on a curve [**Cro2**].

It is worth pointing out that the last two assertions of the previous paragraph are based on cases of Shiho's conjecture on logarithmic extensions of overconvergent $F$-isocrystals (see Section 1.4). That conjecture is a theorem of Tsuzuki for unit-root isocrystals [**Tsz4**] and of Kedlaya for curves [**Ke2**]; it would appear that (by adapting Caro's arguments to a suitable logarithmic setting) a resolution of Shiho's conjecture could be used to establish overholonomicity of arbitrary overconvergent $F$-isocrystals, which then (by appropriate dévissages) should be useful for extending the stability of overholonomicity under the remaining cohomological operations.

[13]This is not to say categorically that such an analogue does not exist! A truly analytic approach to the notion of holonomicity would be quite interesting.

However, these assertions, as well as Shiho's conjecture, remain in the future at the time of this writing.

## 3. $p$-adic Hodge theory

By "$p$-adic Hodge theory", we will mean two things: the study of the interrelated structures on the various cohomologies (notably de Rham and $p$-adic étale) of varieties over $p$-adic fields, and the study of the abstract objects (Galois representations and various parameter modules) arising from these interrelationships. In particular, one central question is to classify $p$-adic Galois representations which are "geometrically interesting", though the meaning of "interesting" has expanded recently (see Section 3.3).

For further discussion of much of this material, we recommend highly Berger's recent survey [**Brg2**].

**3.1. Geometric $p$-adic Hodge theory.** The origin of $p$-adic Hodge theory lies in Grothendieck's proposal of a "mysterious functor" (*foncteur mystérieux*) that would directly relate the $p$-adic étale cohomology and the crystalline cohomology of an algebraic variety over a $p$-adic field. This question in turn originated in the cohomology in degree 1, where it naturally occurs phrased in terms of $p$-divisible groups, and was answered by Fontaine [**Fo1, Fo2**].

Fontaine then constructed [**Fo3, Fo4**] a general setup for approaching Grothendieck's question, which we now briefly introduce. For $K$ a finite extension of $\mathbb{Q}_p$, let $G_K$ denote the absolute Galois group of $K$. By a *$p$-adic representation* of $G_K$, we will always mean a finite dimensional $\mathbb{Q}_p$-vector space $V$ equipped with a continuous $G_K$-action. By a *ring of $p$-adic periods*, we will mean a topological $\mathbb{Q}_p$-algebra $\mathbf{B}$ equipped with a continuous $G_K$-action, such that $\mathbf{B}$ is *$G_K$-regular*: if $b \in \mathbf{B}$ generates a $G_K$-stable $\mathbb{Q}_p$-subspace of $\mathbf{B}$, then $b \in \mathbf{B}^*$. In particular, the fixed ring $\mathbf{B}^{G_K}$ is a field. For $V$ a $p$-adic representation and $\mathbf{B}$ a ring of $p$-adic periods, we define the "period space"

$$D_{\mathbf{B}}(V) = (V \otimes_{\mathbb{Q}_p} \mathbf{B})^{G_K};$$

it is easily shown to be a $\mathbf{B}^{G_K}$-vector space of dimension less than or equal to the $\mathbb{Q}_p$-dimension of $V$. If equality holds, we say $V$ is $\mathbf{B}$-*admissible*.

Fontaine exhibited a number of rings of $p$-adic periods

$$\mathbf{B}_{\mathrm{crys}}, \mathbf{B}_{\mathrm{st}}, \mathbf{B}_{\mathrm{dR}}, \mathbf{B}_{\mathrm{HT}}$$

the subscripts respectively abbreviating "crystalline", "semistable", "de Rham", "Hodge-Tate"; conversely, one abbreviates "$\mathbf{B}_{\mathrm{crys}}$-admissible" to "crystalline" and so on. These conditions get weaker as you move to the right, so every crystalline representation is semistable, and so on. One may insert the condition "potentially semistable" after "semistable", for a representation which becomes semistable upon restriction to $G_{K'}$ for some finite extension $K'$ of $K$. With this insertion, the reverse implications all fail with one exception; see Section 3.2.

Let $X$ be a smooth proper variety over $K$; then the $p$-adic étale cohomology $V = H^i_{\mathrm{et}}(X \times_K \overline{K}, \mathbb{Q}_p)$ is a $p$-adic representation of $G_K$, and the period rings can be used to extract the "hidden structure" on $H^i_{\mathrm{et}}(X \times_K \overline{K}, \mathbb{Q}_p)$. This was first done (not in this framework or terminology) by Tate, who considered the ring $\mathbf{B}_{\mathrm{HT}} = \oplus_{n \in \mathbb{Z}} \mathbb{C}_p(n)$, where $\mathbb{C}_p$ is the completed algebraic closure of $K$ and $(n)$ denotes a twist by the $n$-th power of the cyclotomic character. Any Hodge-Tate representation thus carries a set of numerical invariants: if $V$ is Hodge-Tate, then $V \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ decomposes as a direct sum of copies of various $\mathbb{C}_p(n)$. The collection of these integers $n$ is called the set of *Hodge-Tate weights* of $V$. Tate conjectured that $H^i_{\mathrm{et}}(\mathbb{Q}_p \times_K \overline{K}, X)$ is always Hodge-Tate with Hodge-Tate weights in $\{0, \dots, i\}$, and that the multiplicity of the weight $j$ is equal to the Hodge number $h^{j,i-j} = \dim_K H^j(\Omega^i_{X/K}, X)$. This was established in limited generality by Bloch and Kato [**BK**], and in full by Faltings [**Fa1**].

To describe the mysterious functor, Fontaine found that Hodge-Tate admissibility was not sufficient. Instead, he passed to the class of de Rham representations and conjectured the following:

- the representation $V = H^i_{\mathrm{et}}(X \times_K \overline{K}, \mathbb{Q}_p)$ is always de Rham;
- there is a canonical isomorphism of $D_{\mathrm{dR}}(V)$ (which is a vector space over $\mathbf{B}_{\mathrm{dR}}^{G_K} = K$) with the de Rham cohomology of $X$, under which the Hodge filtration is obtained by a distinguished filtration on $\mathbf{B}_{\mathrm{dR}}$;
- there is a recipe (only specified later; see below) to get back from the de Rham cohomology to the étale cohomology.

This was proved in several stages. First, in case $X$ has good reduction, Fontaine expected an analogous set of statements involving $\mathbf{B}_{\mathrm{crys}}$; these were proved by Fontaine and Messing [**FM**], Faltings [**Fa2**], and later again (using $K$-theoretic techniques) by Nizioł [**N**]. In case $X$ has semistable reduction, Fontaine and Jannsen expected a similar set of statements involving $\mathbf{B}_{\mathrm{st}}$, specifying the recipe to return from de Rham to étale cohomology; this was ultimately established by Tsuji [**Tsj**], drawing on work of numerous authors.[14] Finally, to deduce the de Rham statement, Tsuji actually shows that $V$ is potentially semistable (i.e., semistable upon restriction to a suitable $G_{K'}$) by adding to the mix de Jong's semistable alterations theorem [**dJ1**].

**3.2. Abstract $p$-adic Hodge theory.** The "abstract" aspect of $p$-adic Hodge theory should be thought of as analogizing the study of abstract Hodge structures, variations of Hodge structures, and the like, without direct reference to algebro-geometric objects. Here an interesting interrelationship emerges between $p$-adic Galois representations and the $p$-adic differential equations considered in Section 1.4.

One important result in the abstract theory is the Colmez-Fontaine theorem, which classifies certain $p$-adic Galois representations in terms of simple linear algebra data (thus perhaps justifying the use of the term "Hodge theory" in the phrase "$p$-adic Hodge theory"). To state it, let $K$ be a finite extension of $\mathbb{Q}_p$ with maximal unramified subextension $K_0$. A *$(\phi, N)$-module* is a finite dimensional $K_0$-vector space $D$ equipped with a Frobenius-semilinear action $\phi : D \to D$, and a $K_0$-linear map $N : D \to D$ satisfying $N\phi = p\phi N$. Such an object is *filtered* if it comes with the data of an exhaustive separated descending filtration $\mathrm{Fil}^i$ on $L \otimes_{L_0} V$ by $G$-stable subspaces. (No condition is made concerning compatibility with $\phi$ or $N$.)

If $V$ is a semistable representation of $G_K$, then $(B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ inherits the structure of a filtered $(\phi, N)$-module by deriving the $\phi$-action, $N$-action, and filtration from such structures described explicitly on $B_{\mathrm{st}}$. Moreover, such filtered $(\phi, N)$-modules are distinguished by a certain numerical property. For a filtered $(\phi, N)$-module $D$ of rank 1, define the *Newton number* $t_N(D)$ as the valuation of

[14] Mokrane's MathSciNet review of [**Tsj**] details amply the swarm of results that funnel into Tsuji's work.

any element $a \in K$ defined by the relation $\phi(d) = ad$ for $d \in D$ nonzero. Define the *Hodge number* $t_H(D)$ as the largest $i$ such that $\mathrm{Fil}^i(D) \neq 0$. For a general filtered $(\phi, N)$-module $D$, define $t_N(D) = t_N(\wedge^{\dim(D)} D)$ and $t_H(D) = t_H(\wedge^{\dim(D)} D)$.

THEOREM 3.1. *A filtered $(\phi, N)$-module $D$ arises from a semistable representation if and only if:*

(a) $t_N(D) = t_H(D)$*;*

(b) *for any $(\phi, N)$-submodule $D'$ of $D$ equipped with the induced filtration, $t_N(D') \geq t_H(D')$.*

To put this in the formalism of semistability of vector bundles, if one defines the "degree" of $D$ to be $t_H(D) - t_N(D)$, then $D$ arises from a semistable representation if and only if it is "semistable of slope 0".

Theorem 3.1 was originally proved by Colmez-Fontaine [**CF**] by an ingenious but technically difficult[15] argument. A different proof was given by Berger [**Brg3**] by working with $(\phi, \Gamma)$-modules[16] over the Robba ring, and at one point invoking Kedlaya's slope filtration theorem (see Section 1.4); a variant of this argument in the crystalline case was given by Kisin [**Ks2**], which as an offshoot established some classification results for $p$-divisible groups and finite flat group schemes conjectured by Breuil (unpublished).

Another key result is Fontaine's conjecture that every de Rham representation is potentially semistable. (Recall that potential semistability for representations arising from étale cohomology was established by Tsuji; see Section 3.1.) This conjecture was first proved by Berger [**Brg1**] again using $(\phi, \Gamma)$-modules over the Robba ring; this time the key input from that theory is precisely Theorem 1.1. Subsequently, proofs within the "Fontaine context" were given by Colmez [**Cmz2**] and Fontaine [**Fo5**].

An additional direction, which has some relevance for applications[17] in number theory, is the relationship between linear algebraic descriptions of $p$-adic representations and Galois cohomology. This was first described by Herr [**Her1, Her2**], who demonstrated its utility by recovering some valuable results of Tate, such as local duality. In a different direction, Marmora [**Mar**] has described a relationship between Swan conductors of a potentially semistable representation and "irregularity" of filtered modules; this is an mixed-characteristic analogue of the Grothendieck-Ogg-Shafarevich formula in rigid cohomology (see Section 1.3).

**3.3. Nongeometric representations and a Langlands correspondence.** In recent years, the subclass among $p$-adic Galois representations of those considered "geometrically interesting" has been significantly enlarged. The old definition would have restricted to those which are potentially semistable (or de Rham, which is the same; see Section 3.2), as those are the ones which can occur within $p$-adic étale cohomology. However, the theory of modular forms suggests that a bigger class

[15] To be fair, the same characterization could reasonably be made of [**Ke3**], which underlies Berger's proof; one counterargument is that the results of [**Ke3**] are applicable a bit more broadly.

[16] A $(\phi, \Gamma)$-module is an algebraic object that describes a $p$-adic representation by replacing the complicated Galois action with a simple action on a more complicated ring. Again, see [**Brg2**] for more on the utility of such objects.

[17] Example: the upcoming Brandeis thesis of Seunghwan Chang will apply these ideas to the formulation of variants of Serre's conjecture on modularity of Galois representations.

should be considered, including certain "interpolations" among geometric representations.

Specifically, to address a qualitative[18] version of the Gouvêa-Mazur conjecture [**GM**] on $p$-adic variation of modular forms, Coleman [**Cmn2**] introduced the class of *overconvergent*[19] *modular forms*. Coleman and Mazur [**CoM**] demonstrated that these were attached to (global) Galois representations naturally parametrized by a rigid analytic curve (the *eigencurve*). Kisin [**Ks1**] showed that of these, only the representations attached to classical modular forms have local representations which are potentially semistable.

In order to understand the local representations arising on the eigencurve, e.g., to prove theorems[20] on the modularity of Galois representations in the vein of Taylor and Wiles [**Wi, TW**], one would like to describe a classification of $p$-adic representations in the spirit of the local Langlands correspondence of Harris and Taylor [**HT**] and Henniart [**Hen**] (also see [**Cry**]) for $\ell$-adic representations. Since there are many more $p$-adic representations, the $p$-adic correspondence will necessarily have a different flavor: the "automorphic" representations of $\mathrm{GL}_n(K)$, for $K$ a $p$-adic field, corresponding to $n$-dimensional representations of $G_K$ should be *infinite-dimensional* vector spaces. The appropriate such representations appear to be the locally analytic representations of Schneider and Teitelbaum [**ST1, ST2, ST3, ST4**].

Almost all work in this direction so far has focused on the case of $\mathrm{GL}_2$. Some initial evidence in this case was provided by the work of Breuil and Mézard [**BM**], and by subsequent work of Breuil [**Br1, Br2**]. This has led Breuil to a series of predictions about 2-dimensional representations of $G_K$ and about integral structures on said representations, which appear to relate to Banach lattices on the automorphic sides. Some of these predictions in the case of crystalline representations have now been checked by Berger and Breuil [**BB1, BB2**].

As promised above, further investigation has begun to suggest new classes of representations which are meaningful in the Langlands correspondence. One such class is the class of *trianguline* representations introduced by Colmez [**Cmz3**]. These are 2-dimensional representations of $G_K$ whose $(\phi, \Gamma)$-modules over the Robba ring can be written as extensions of one rank 1 module by another. (Note that the two rank 1 modules do not in general correspond to representations; this characterization thus lies firmly within the theory of Frobenius modules over the Robba ring. See Section 1.4.) Colmez proposes a conjectural correspondence between trianguline representations and unitary principal series representations of $\mathrm{GL}_2(K)$. For these representations, Breuil's conjecture on the mod $p$ reduction has been verified by Berger [**Brg4**].

**3.4. Towards nonabelian $p$-adic Hodge theory.** We end with a brief mention of some results in the direction of "nonabelian $p$-adic Hodge theory". If one thinks of "abelian" Hodge theory, $p$-adic or otherwise, as the study of extra structures on the abelianization of the fundamental group of a variety, then nonabelian

[18]The quantitative version of the Gouvêa-Mazur conjecture is actually incorrect as stated [**BC**].

[19]This word will look familiar; it refers to the fact that roughly speaking, these are defined as sections of a sheaf on a dagger space associated to a modular curve. The "roughly" is because that sheaf may have to be raised to a non-integral power; Iovita and Stevens are working on several ways to render Coleman's *ad hoc* workaround for this more systematic.

[20]This has already begun: Kisin has given a strengthening [**Ks3**] of his modularity results using the work of Berger and Breuil.

Hodge theory should be the study of extra structures on possibly nonabelian quotients of the fundamental group. For example, in ordinary Hodge theory, Simpson [**Si**] gives an analogue of the Hodge decomposition for the cohomology of a variety with coefficients in a local system (representation of the fundamental group), using Higgs bundles.

One source of inspiration for nonabelian Hodge theory was Deligne's tract [**De**] on the projective line minus three points; it gives a natural Tannakian framework for considering the Hodge theory of *unipotent* quotients of the fundamental group. In particular, there is a realization of this framework corresponding to crystalline cohomology, giving rise to a crystalline fundamental group. This generalizes pretty broadly, to suitable log schemes over a field of positive characteristic [**ClS, Sh1, Sh2**].

A more robust foundation for nonabelian Hodge theory comes from rational homotopy theory, as in the work of Katzarkov-Pantev-Toen [**KPT**]. This for starters gives a more uniform construction of crystalline fundamental groups [**HK**]. That in turn should fit into a fuller nonabelian $p$-adic Hodge theory parallel to that of [**KPT**]; results in this direction have been obtained by Olsson [**Ol2, Ol3**].

We cannot conclude without pointing out that something as seemingly abstruse nonabelian $p$-adic Hodge theory may have concrete applications to Diophantine equations! Kim [**Km**] has suggested a nonabelian generalization of Chabauty's method [**Ch**] for bounding the number of, and in some cases determining the exact set of, rational points on a curve over a number field. Chabauty's method has been rendered practical by a series of refinements [**Cmn1, Fl, FW**], but usually only works when the Mordell-Weil group of the Jacobian has rank less than the genus of the curve (so that the group lies within a closed $p$-adic analytic subvariety of the Jacobian). It is hoped that the nonabelian version, which would take place on a higher Albanese variety [**Hai**] instead of the Jacobian, may yield additional practical results in cases where this "Chabauty condition" is not satisfied.

## References


[A] Y. André, Filtrations de type Hasse-Arf et monodromie $p$-adique, *Invent. Math.* **148** (2002), 285–317.

[Brg1] L. Berger, Représentations $p$-adiques et équations différentielles, *Invent. Math.* **148** (2002), 219–284.

[Brg2] ———, An introduction to the theory of $p$-adic representations, in *Geometric aspects of Dwork theory*, Volume I, de Gruyter, Berlin, 2004, 255–292.

[Brg3] ———, Équations différentielles et $(\phi, N)$-modules filtrés, arXiv preprint `math.NT/0406601`.

[Brg4] ———, Représentations modulaires de $\mathrm{GL}_2(\mathbf{Q}_p)$ et représentations galoisiennes de dimension 2, arXiv preprint `math.NT/0510090`.

[BB1] L. Berger and C. Breuil, Représentations cristallines irréductibles de $\mathrm{GL}_2(\mathbb{Q}_p)$, arXiv preprint `math.NT/0410053`.

[BB2] ———, Sur la réduction des représentations cristallines de dimension 2 en poids moyens, arXiv preprint `math.NT/0504388`.

[Brt1] P. Berthelot, *Cohomologie cristalline des schémas de caractéristique* $p > 0$, Lecture Notes in Math. 407, Springer-Verlag, Berlin, 1974.

[Brt2] ———, Géométrie rigide et cohomologie des variétés algébriques de caractéristique $p$, in Introduction aux cohomologies $p$-adiques, *Mém. Soc. Math. France* **23** (1986), 7–32.

[Brt3] ———, $\mathcal{D}$-modules arithmétiques. I. Opérateurs différentiels de niveau fini, *Ann. Sci. École Norm. Sup.* **29** (1996), 185–272.

[Brt4] ———, Finitude et pureté cohomologique en cohomologie rigide (with an appendix in English by A.J. de Jong), *Invent. Math.* **128** (1997), 329–377.

[Brt5] ______, Dualité de Poincaré et formule de Künneth en cohomologie rigide, *C.R. Acad. Sci. Paris Sér. I Math.* **325** (1997), 493–498.

[Brt6] ______, Altérations de variétés algébriques (d'après A.J. de Jong), Sém. Bourbaki 1995/96, *Astérisque* **241** (1997), 275–311.

[Brt7] ______, $\mathcal{D}$-modules arithmétiques. II. Descente par Frobenius, *Mém. Soc. Math. France* **81** (2000).

[Brt8] ______, Introduction à la théorie arithmétique des $\mathcal{D}$-modules, Cohomologies $p$-adiques et applications arithmétiques, II, *Astérisque* **279** (2002), 1–80.

[BO1] P. Berthelot and A. Ogus, *Notes on crystalline cohomology*, Princeton Univ. Press, Princeton, 1978.

[BO2] ______, $F$-isocrystals and de Rham cohomology. I, *Invent. Math.* **72** (1983), 159–199.

[BE] S. Bloch and H. Esnault, Homology for irregular connections, *J. Théor. Nombres Bordeaux* **16** (2004), 357–371.

[BK] S. Bloch and K. Kato, $p$-adic étale cohomology, *Publ. Math. IHÉS* **63** (1986), 107–152.

[BGR] S. Bosch, U. Güntzer, and R. Remmert, *Non-archimedean analysis*, Grundlehren der Math. Wissenschaften 261, Springer-Verlag, Berlin, 1984.

[Br1] C. Breuil, Sur quelques représentations modulaires et $p$-adiques de $\mathrm{GL}_2(\mathbf{Q}_p)$. I, *Compos. Math.* **138** (2003), 165–188.

[Br2] ______, Sur quelques représentations modulaires et $p$-adiques de $\mathrm{GL}_2(\mathbf{Q}_p)$. II, *J. Inst. Math. Jussieu* **2** (2003), 23–58.

[BM] C. Breuil and A. Mézard, Multiplicités modulaires et représentations de $\mathrm{GL}_2(\mathbb{Z}_p)$ et de $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ en $l = p$, with an appendix by G. Henniart, *Duke Math. J.* **115** (2002), 205–310.

[BC] K. Buzzard and F. Calegari, A counterexample to the Gouvêa-Mazur conjecture, *C.R. Math. Acad. Sci. Paris* **338** (2004), 751–753.

[Cry] H. Carayol, Preuve de la conjecture de Langlands locale for $\mathrm{GL}_n$: travaux de Harris-Taylor et Henniart, Séminaire Bourbaki, Vol. 1998/1999, *Astérisque* **266** (2000), 191–243.

[Cro1] D. Caro, $\mathcal{D}$-modules arithmétiques surcohérents. Application aux fonctions $L$, *Ann. Inst. Fourier (Grenoble)* **54** (2004), 1943–1996.

[Cro2] ______, Fonctions $L$ associées aux $\mathcal{D}$-modules arithmétiques. Cas des courbes, to appear in *Compos. Math.*; preprint available at `http://www.maths.dur.ac.uk/~dma1dc/`.

[Cro2] ______, Dévissages des $F$-complexes de $\mathcal{D}$-modules arithmétiques en $F$-isocristaux surconvergents, arXiv preprint `math.AG/0503642`.

[Cro3] ______, Sur la compatibilité à Frobenius de l'isomorphisme de dualité relative, arXiv preprint `math.AG/0509448`.

[Cro4] ______, $\mathcal{D}$-modules arithmétiques associés aux isocristaux surconvergents. Cas lisse, arXiv preprint `math.AG/0510422`.

[Cro5] ______, $\mathcal{D}$-modules arithmétiques surholonomes, preprint available at `http://www.maths.dur.ac.uk/~dma1dc/`.

[Ch] C. Chabauty, Sur les points rationnels des courbes algébriques de genre supérieur à l'unité, *C.R. Acad. Sci. Paris* **212** (1941), 882–885.

[ClS] B. Chiarellotto and B. le Stum, $F$-isocristaux unipotents, *Compos. Math.* **116** (1999), 81–110.

[CT] B. Chiarellotto and N. Tsuzuki, Cohomological descent of rigid cohomology for étale coverings, *Rend. Sem. Math. Univ. Padova* **109** (2003), 63–215.

[ChM1] G. Christol and Z. Mebkhout, Sur le théorème de l'indice des équations différentielles $p$-adiques. I, *Ann. Inst. Fourier (Grenoble)* **43** (1993), 1545–1574.

[ChM2] ______, Sur le théorème de l'indice des équations différentielles $p$-adiques. II, *Ann. of Math.* **146** (1997), 345–410.

[ChM3] ______, Sur le théorème de l'indice des équations différentielles $p$-adiques. III, *Ann. of Math.* **151** (2000), 385–457.

[ChM4] ______, Sur le théorème de l'indice des équations différentielles $p$-adiques. IV, *Invent. Math.* **143** (2001), 629–672.

[Cmn1] R. Coleman, Effective Chabauty, *Duke Math. J.* **52** (1985), 765–770.

[Cmn2] ______, $p$-adic Banach spaces and families of modular forms, *Invent. Math.* **127** (1997), 417–479.

[CI1] R. Coleman and A. Iovita, The Frobenius and monodromy operators for curves and abelian varieties, *Duke Math. J.* **97** (1999), 171–215.

[CI2] ______, Revealing hidden structures, preprint available at `http://www.mathstat.concordia.ca/faculty/iovita/`.

[CoM] R. Coleman and B. Mazur, The eigencurve, in *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, London Math. Soc. Lecture Note Series 254, Cambridge Univ. Press, Cambridge, 1998, 1–113.

[Cmz1] P. Colmez, Les conjectures de monodromie $p$-adiques, Séminaire Bourbaki, Vol. 2001/2002, *Astérisque* **290** (2003), 53–101.

[Cmz2] ______, Espaces Vectoriels de dimension finie et représentations de de Rham, preprint available at `http://www.math.jussieu.fr/~colmez/`.

[Cmz3] ______, Série principale unitaire pour $\mathrm{GL}_2(\mathbb{Q}_p)$ et représentations triangulines de dimension 2, preprint available at `http://www.math.jussieu.fr/~colmez/`.

[CF] P. Colmez and J.-M. Fontaine, Construction des représentations $p$-adiques semi-stables, *Invent. Math.* **140** (2000), 1–43.

[Cr1] R. Crew, $F$-isocrystals and their monodromy groups, *Ann. Sci. École Norm. Sup.* **25** (1992), 429–464.

[Cr2] ______, Finiteness theorems for the cohomology of an overconvergent isocrystal on a curve, *Ann. Sci. École Norm. Sup.* **31** (1998), 717–763.

[Cr3] ______, Canonical extensions, irregularities, and the Swan conductor, *Math. Ann.* **316** (2000), 19–37.

[dJ1] A.J. de Jong, Smoothness, semi-stability and alterations, *Publ. Math. IHÉS* **83** (1996), 51–93.

[dJ2] ______, Barsotti-Tate groups and crystals, Proceedings of the ICM (Berlin, 1998), *Doc. Math.* Extra Vol. II (1998), 301–333.

[DLZ] C. Davis, A. Langer, and T. Zink, An overconvergent de Rham-Witt complex, in preparation.

[De] P. Deligne, Le groupe fondamental de la droite projective moins trois points, in *Galois groups over $\mathbb{Q}$ (Berkeley, CA, 1987)*, Math. Sci. Res. Inst. Publ. 16, Springer, New York, 1989, 79–297.

[ElS] J.-Y. Étesse and B. le Stum, Fonctions $L$ associées aux isocristaux surconvergents. I. Interprétation cohomologique, *Math. Ann.* **296** (1993), 557–576.

[Fa1] G. Faltings, $p$-adic Hodge theory, *J. Amer. Math. Soc.* **1** (1988), 255–299.

[Fa2] ______, Crystalline cohomology and $p$-adic Galois-representations, in *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, Johns Hopkins Univ. Press, Baltimore, 1989, 25–80.

[Fl] E.V. Flynn, A flexible method for applying Chabauty's theorem, *Compos. Math.* **105** (1997), 79–94.

[FW] E.V. Flynn and J.L. Wetherell, Finding rational points on bielliptic genus 2 curves, *Manuscripta Math.* **100** (1999), 519–533.

[Fo1] J.-M. Fontaine, Modules galoisiens, modules filtrés et anneaux de Barsotti-Tate, Journées de Géométrie Algébrique de Rennes (Rennes, 1978), Vol. III, *Astérisque* **65** (1979), 3–80.

[Fo2] ______, Sur certains types de représentations $p$-adiques du groupe de Galois d'un corps local; construction d'un anneau de Barsotti-Tate, *Annals of Math.* **115** (1982), 529–577.

[Fo3] ______, Le corps des périodes $p$-adiques, in Périodes $p$-adiques (Bures-sur-Yvette, 1988), *Astérisque* **223** (1994), 59–111.

[Fo4] ______, Représentations $p$-adiques semi-stables, in Périodes $p$-adiques (Bures-sur-Yvette, 1988), *Astérisque* **223** (1994), 113–184.

[Fo5] ______, Représentations de de Rham et représentations semi-stables, Orsay prépublication 2004-12, available at `http://www.math.u-psud.fr/~biblio/pub/2004/`.

[FM] J.-M. Fontaine and W. Messing, $p$-adic periods and $p$-adic étale cohomology, in *Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985)*, Contemp. Math. 67, Amer. Math. Soc., Providence, 1987, 179–207.

[FvdP] J. Fresnel and M. van der Put, *Rigid analytic geometry and its applications*, Progress in Math. 218, Birkhäuser, Boston, 2004.

[GK1] E. Grosse-Klönne, Rigid analytic spaces with overconvergent structure sheaf, *J. reine angew. Math.* **519** (2000), 73–95.

[GK2] ______, Finiteness of de Rham cohomology in rigid analysis, *Duke Math. J.* **113** (2002), 57–91.

[GM] F. Gouvêa and B. Mazur, Families of modular eigenforms, *Math. Comp.* **58** (1992), 793–805.

[Hai] R.M. Hain, Higher Albanese manifolds, in *Hodge theory (Sant Cugat, 1985)*, Lecture Notes in Math. 1246, Springer, Berlin, 1987, 84–91.

[HK] R.M. Hain and M. Kim, A De Rham-Witt approach to crystalline rational homotopy theory, *Compos. Math.* **140** (2004), 1245–1276.

[HT] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties* (with an appendix by V.G. Berkovich), Annals of Math. Studies 151, Princeton Univ. Press, Princeton, 2001.

[Har1] R. Hartshorne, Algebraic de Rham cohomology, *Manuscripta Math.* **7** (1972), 125–140.

[Har2] ———, On the de Rham cohomology of algebraic varieties, *Publ. Math. IHÉS* **45** (1975), 5–99.

[Hen] G. Henniart, Une preuve simple des conjectures de Langlands pour $\mathrm{GL}(n)$ sur un corps $p$-adique, *Invent. Math.* **139** (2000), 439–455.

[Her1] L. Herr, Sur la cohomologie galoisienne des corps $p$-adiques, *Bull. Soc. Math. France* **126** (1998), 563–600.

[Her2] ———, Une approche nouvelle de la dualité locale de Tate, *Math. Ann.* **320** (2001), 307–337.

[HL] M. Herrera and D. Lieberman, Duality and the de Rham cohomology of infinitesimal neighborhoods, *Invent. Math.* **13** (1971), 97–124.

[Hu1] C. Huyghe, Construction et étude de la transformation de Fourier pour les $\mathcal{D}$-modules arithmétiques, Thèse de Doctorat, Université de Rennes 1, 1995.

[Hu2] ———, Thèoréme d'acyclicité pour les $\mathcal{D}^\dagger_{\mathbb{Q}}$-modules sur l'espace projectif, *C.R. Acad. Sci. Paris* **321** (1995), 453–455.

[Hu3] ———, Interprétation géométrique sur l'espace projectif des $A_N(K)^\dagger$-modules cohérents, *C.R. Acad. Sci. Paris* **321** (1995), 587–590.

[Hu4] ———, Transformation de Fourier des $\mathcal{D}^\dagger_{\mathcal{X},\mathbb{Q}}$-modules, *C.R. Acad. Sci. Paris* **321** (1995), 759–762.

[Hu5] ———, $\mathcal{D}^\dagger$-affinité de de l'espace projectif (with an appendix by P. Berthelot), *Compos. Math.* **108** (1997), 277–318.

[Hu6] ———, $\mathcal{D}^\dagger(\infty)$-affinité des schémas projectifs, *Ann. Inst. Fourier (Grenoble)* **48** (1998), 913–956.

[Hu7] ——— (as C. Noot-Huyghe), Transformation de Fourier des $\mathcal{D}$-modules arithmétiques I, in *Geometric aspects of Dwork theory*, Volume II, de Gruyter, Berlin, 2004, 857–907.

[I] L. Illusie, *Crystalline cohomology*, in *Motives*, Proc. Sympos. Pure Math., vol. 55, part 1, Amer. Math. Soc. Providence, RI, 1994, 43–70.

[KPT] L. Katzarkov, T. Pantev, and B. Toen, Schematic homotopy types and non-abelian Hodge theory I: The Hodge decomposition, arXiv preprint `math.AG/0107129`.

[Ke1] K.S. Kedlaya, Computing zeta functions via $p$-adic cohomology, in *Algorithmic number theory (ANTS VI)*, Lecture Notes in Comp. Sci. 3076, Springer-Verlag, Berlin, 2004, 1–17.

[Ke2] ———, Semistable reduction for overconvergent $F$-isocrystals on a curve, *Math. Res. Lett.* **10** (2003), 151–159.

[Ke3] ———, A $p$-adic local monodromy theorem, *Ann. of Math.* **160** (2004), 93–184.

[Ke4] ———, Full faithfulness for overconvergent $F$-isocrystals, in *Geometric aspects of Dwork theory*, Volume II, de Gruyter, Berlin, 2004, 819–835.

[Ke5] ———, More étale covers of affine spaces in positive characteristic, *J. Alg. Geom.* **14** (2005), 187–192.

[Ke6] ———, Slope filtrations revisited, *Doc. Math.* **10** (2005), 447–525.

[Ke6] ———, Finiteness of rigid cohomology with coefficients, *Duke Math. J.*, to appear; arXiv preprint `math.AG/0208027` (version of 2 Nov 2005).

[Ke7] ———, Fourier transforms and $p$-adic "Weil II", arXiv preprint `math.NT/0210149` (version of 20 Jul 2005).

[Ke8] ———, Semistable reduction for overconvergent $F$-isocrystals, I: Unipotence and logarithmic extensions, arXiv preprint `math.NT/0405069` (version of 24 Jul 2005).

[Km] M. Kim, The motivic fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and the theorem of Siegel, arXiv preprint `math.NT/0409546` (version of 23 Sep 2004).

[Ks1] M. Kisin, Overconvergent modular forms and the Fontaine-Mazur conjecture, *Invent. Math.* **153** (2003), 373–454.

[Ks2] ———, Crystalline representations and $F$-crystals, preprint available at `http://www.math.uchicago.edu/~kisin`.

[Ks3] ———, Modularity of some geometric Galois representations, preprint available at `http://www.math.uchicago.edu/~kisin`.

[Kl] S. Kleiman, Algebraic cycles and the Weil conjectures, in *Dix exposés sur la cohomologie des schémas*, Masson, Paris and North-Holland, Amsterdam, 1968, 359–386.

[LW] A.G.B. Lauder and D. Wan, Counting rational points over finite fields of small characteristic, preprint available at `http://www.math.uci.edu/~dwan/`.

[LS] B. Le Stum, *Rigid Cohomology*, Cambridge Univ. Press, 2007.

[Mar] A. Marmora, Irrégularité et conducteur de Swan $p$-adiques, *Doc. Math.* **9** (2004), 413–433.

[Mat] S. Matsuda, Katz correspondence for quasi-unipotent overconvergent isocrystals, *Comp. Math.* **134** (2002), 1–34.

[MST] B. Mazur, W. Stein, and J. Tate, Computation of $p$-adic heights and log convergence, preprint available at `http://modular.ucsd.edu/`.

[Meb1] Z. Mebkhout, Sur le théorème de finitude de la cohomologie $p$-adique d'une variété affine non singulière, *Amer. J. Math.* **119** (1997), 1027–1081.

[Meb2] ———, Analogue $p$-adique du Théorème de Turrittin et le Théoréme de la monodromie $p$-adique, *Invent. Math.* **148** (2002), 319–351.

[MN] Z. Mebkhout and L. Narváez-Macarro, Sur les coefficients de Grothendieck-de Rham des variétés algébriques, in *$p$-adic analysis (Trento, 1989)*, Lecture Notes in Math. 1454, Springer, Berlin, 1990.

[Mer] D. Meredith, Weak formal schemes, *Nagoya Math. J.* **45** (1971), 1–38.

[Mok] A. Mokrane, La suite spectrale des poids en cohomologie de Hyodo-Kato, *Duke Math. J.* **72** (1993), 301–337.

[Mon1] P. Monsky, Formal cohomology. II. The cohomology sequence of a pair, *Ann. of Math.* **88** (1968), 218–238.

[Mon2] ———, Formal cohomology. III. Fixed points theorems, *Ann. of Math.* **93** (1971), 315–343.

[MW] P. Monsky and G. Washnitzer, Formal cohomology. I, *Ann. of Math.* **88** (1968), 181–217.

[N] W. Nizioł, Crystalline conjecture via $K$-theory, *Ann. Sci. École Norm. Sup.* **31** (1998), 659–681.

[Og1] A. Ogus, $F$-isocrystals and de Rham cohomology. II. Convergent isocrystals, *Duke Math. J.* **51** (1984), 765–850.

[Og2] ———, The convergent topos in characteristic $p$, in *The Grothendieck Festschrift*, volume III, Progress in Math. 88, Birkhäuser, Boston, 1990.

[Ol1] M. Olsson, Crystalline cohomology of algebraic stacks and Hyodo-Kato cohomology, preprint available at `http://www.ma.utexas.edu/~molsson`.

[Ol2] ———, $F$-isocrystals and homotopy types, preprint available at `http://www.ma.utexas.edu/~molsson`.

[Ol3] ———, Towards non-abelian $p$-adic Hodge theory in the good reduction case, preprint available at `http://www.ma.utexas.edu/~molsson`.

[P] D. Petrequin, Classes de Chern et classes de cycle en cohomologie rigide, *Bull. Soc. Math. France* **131** (2003), 59–121.

[ST1] P. Schneider and J. Teitelbaum, $U(\mathfrak{g})$-finite locally analytic representations (with an appendix by D. Prasad), *Representation Theory* **5** (2001), 111-128.

[ST2] ———, Locally analytic representations and $p$-adic representation theory, with applications to $\mathrm{GL}_2$, *J. Amer. Math. Soc.* **15** (2002), 443–468.

[ST3] ———, Banach space representations and Iwasawa theory, *Israel J. Math.* **127** (2002), 359–380.

[ST4] ———, Algebras of $p$-adic distributions and admissible representations, *Invent. Math.* **153** (2003), 145–196.

[Sh1] A. Shiho, Crystalline fundamental groups. I. Isocrystals on log crystalline site and log convergent site, *J. Math. Sci. Univ. Tokyo* **7** (2000), 509–656.

[Sh2] ———, Crystalline fundamental groups. II. Log convergent cohomology and rigid cohomology, *J. Math. Sci. Univ. Tokyo* **9** (2002), 1–163.

[Si] C. Simpson, Higgs bundles and local systems, *Publ. Math. IHÉS* **75** (1992), 5–95.

[TW] R. Taylor and A. Wiles, Ring-theoretic properties of certain Hecke algebras, *Annals of Math.* **141** (1995), 553–572.

[Tsj] T. Tsuji, $p$-adic étale cohomology and crystalline cohomology in the semi-stable reduction case, *Invent. Math.* **137** (1999), 233–411.

[Tsz1] N. Tsuzuki, Finite local monodromy of overconvergent unit-root $F$-isocrystals on a curve, *Amer. J. Math.* **120** (1998), 1165–1190.

[Tsz2] ———, The local index and the Swan conductor, *Comp. Math.* **111** (1998), 245–288.

[Tsz3] ———, Slope filtration of quasi-unipotent overconvergent $F$-isocrystals, *Ann. Inst. Fourier (Grenoble)* **48** (1998), 379–412.

[Tsz4] ———, Morphisms of $F$-isocrystals and the finite monodromy theorem for unit-root $F$-isocrystals, *Duke Math. J.* **111** (2002), 385–418.

[Tsz5] ———, Cohomological descent of rigid cohomology for proper coverings, *Invent. Math.* **151** (2003), 101–133.

[Tsz6] ———, Cohomological descent in rigid cohomology, in *Geometric aspects of Dwork theory*, Volume II, de Gruyter, Berlin, 2004, 931–981.

[vdP] M. van der Put, The cohomology of Monsky and Washnitzer, in Introduction aux cohomologies $p$-adiques (Luminy, 1984), *Mém. Soc. Math. France* **23** (1986), 33–59.

[V1] A. Virrion, Théorèmes de dualité locale et globale dans la théorie arithmétique des $\mathcal{D}$-modules, Thèse de Doctorat, Université de Rennes 1, 1995.

[V2] ———, Théorème de bidualité et caractérisation des $F$-$\mathcal{D}^\dagger_{\mathcal{X},\mathbb{Q}}$-modules holonomes, *C.R. Acad. Sci. Paris* **319** (1994), 1283–1286.

[V3] ———, Théorème de dualité relative pour les $\mathcal{D}$-modules arithmétiques, *C.R. Acad. Sci. Paris* **321** (1995), 751–754.

[V4] ———, Dualité locale et holonomie pour les $\mathcal{D}$-modules arithmétiques, *Bull. Soc. Math. France* **128** (2000), 1–68.

[V5] ———, Trace et dualité relative pour les $\mathcal{D}$-modules arithmétiques, in *Geometric aspects of Dwork theory*, Volume II, de Gruyter, Berlin, 2004, 1039–1112.

[VZ] J.F. Voloch and M. Zarzar, Algebraic geometric codes on surfaces, preprint available at `http://www.ma.utexas.edu/users/zarzar/`.

[Wa] D. Wan, Algorithmic theory of zeta functions over finite fields, preprint available at `http://www.math.uci.edu/~dwan/`.

[Wi] A. Wiles, Modular elliptic curves and Fermat's last theorem, *Annals of Math.* **141** (1995), 443–551.

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